J. Fluid Mech. (1990), *vol.* 218, *pp.* 143–162
Printed in Great Britain

A generalization of the impulse and virial theorems with an application to bubble oscillations

By HASAN N. OGUZ AND ANDREA PROSPERETTI
Department of Mechanical Engineering, The Johns Hopkins University, Baltimore, MD 21218, USA

(Received 11 August 1989 and in revised form 29 January 1990)

In the first part of the paper it is shown that the impulse and virial theorems of inviscid incompressible fluid mechanics are special cases of a more general theorem from which an infinity of relations can be obtained. Depending on the problem, only a finite number of these relations may be independent. An application of these results is in the approximate study of the hydrodynamic interaction of bodies. As an example, in the second part of the paper, the case of two freely translating, nonlinearly pulsating bubbles is considered. It is found that in certain parameter ranges the force between the bubbles has a sign opposite to what would be expected on the basis of the linear theory of Bjerknes forces.

1. Introduction

Consider N closed material surfaces S_i in a finite or infinite region Ω occupied by a perfect fluid in irrotational motion. Then Blake & Cerone (1982) proved the following relation:

$$\frac{\mathrm{d}}{\mathrm{d}t}\sum_{i=1}^{N}\int_{S_i}\phi\boldsymbol{n}\,\mathrm{d}S_i=\int_B[\tfrac{1}{2}\boldsymbol{u}\cdot\boldsymbol{u}\boldsymbol{n}-(\boldsymbol{n}\cdot\boldsymbol{u})\,\boldsymbol{u}]\,\mathrm{d}S_\mathrm{B}. \tag{1.1}$$

In this relation ϕ is the velocity potential, $\boldsymbol{u}=\nabla\phi$ is the velocity field, and $\boldsymbol{n}$ is the unit normal directed away from the fluid. By B we denote all the material surfaces bounding the region Ω other than $S_1, S_2, \ldots$ and the surface at infinity. In the absence of any boundary at a finite distance from the bodies, the right-hand side vanishes and this relation proves the time independence of the sum of integrals in the left-hand side, which is identified with the *impulse* of the fluid (or, more precisely, with the impulse divided by the density). Benjamin & Ellis (1966) and Blake and co-workers (Blake 1983, 1988; Blake & Cerone 1982; Blake & Gibson 1981, 1987; Blake, Taib & Doherty 1986) have given a number of examples of the application of this theorem in bubble dynamics.

Another integral theorem, only valid for an infinite region Ω, has been proven by Benjamin (1987) and, in a different way, by Longuet-Higgins (1989). In the previous notation this theorem may be written

$$\frac{\mathrm{d}}{\mathrm{d}t}\sum_{i=1}^{N}\rho\int_{S_i}-\phi\boldsymbol{x}\cdot\boldsymbol{n}\,\mathrm{d}S_i=-5E_\mathrm{K}+\sum_{i=1}^{N}\int_{S_i}(p-p_\infty)\,(\boldsymbol{x}\cdot\boldsymbol{n})\,\mathrm{d}S_i, \tag{1.2}$$

where E_K is the kinetic energy of the fluid, p is the pressure, ρ is the density and p_∞ the ambient pressure. The sum of integrals in the left-hand side is the *virial* of the motion.

In the present paper we shall generalize these results in two directions. First, we shall show that they are special cases of a much more general theorem. Secondly, we shall derive corresponding relations valid for each particular surface S_i, or portion thereof, rather than for the sums appearing in the left-hand sides of (1.1) and (1.2). In this sense our general theorem is 'local', while (1.1) and (1.2) are 'global'.

We believe that our results are interesting in two complementary ways. First, they can lead to the discovery of conservation properties and thus offer an insight into the structure of the theory. Secondly, they can be useful in setting up systematic approximation schemes for specific problems, as will be shown in §5 for the case of two interacting bubbles. In view of the remarkable accuracy with which Blake and co-workers have been able to estimate the direction of the jet of a collapsing bubble by means of the impulse theorem (1.1), it may be expected that such an approach could be fruitful. Other examples of the usefulness of integral theorems have recently been given by Benjamin (1989) and Longuet-Higgins (1989) who have demonstrated their use in the context of slightly non-spherical oscillating bubbles.

2. Local theorems

The most general form of our 'local' integral theorem is rather complex. For this reason, in order to illustrate the method of derivation, it is useful to consider some simple special cases first.

For an inviscid flow and in the absence of body forces the Bernoulli integral can be written as

$$\frac{d\phi}{dt}-\tfrac{1}{2}\boldsymbol{u}\cdot\boldsymbol{u}+p' = 0, \tag{2.1}$$

where d/dt denotes the material derivative and

$$p' = \frac{p-p_\infty}{\rho},$$

with ρ the liquid density and p_∞ an integration constant that can depend on time. For a material surface S and any scalar, vector, or tensor quantity f we have

$$\frac{d}{dt}\int_S f\, dS = \int_S \frac{df}{dt}\, dS - \int_S f(\boldsymbol{n}\cdot\boldsymbol{\nabla})(\boldsymbol{u}\cdot\boldsymbol{n})\, dS, \tag{2.2}$$

as directly follows from

$$\frac{1}{\delta S}\frac{d}{dt}\delta S = -(\boldsymbol{n}\cdot\boldsymbol{\nabla})(\boldsymbol{u}\cdot\boldsymbol{n}) \tag{2.3}$$

(see e.g. Aris 1962 or Kemmer 1977 for a detailed derivation, or Prosperetti 1979 for a simpler approach). Upon integration of (2.1) over such a material surface S and use of this relation we obtain

$$\frac{d}{dt}\int_S \phi\, dS = -\int_S \phi(\boldsymbol{n}\cdot\boldsymbol{\nabla})(\boldsymbol{u}\cdot\boldsymbol{n})\, dS + \frac{1}{2}\int_S \boldsymbol{u}\cdot\boldsymbol{u}\, dS - \int_S p'\, dS. \tag{2.4}$$

This is a special case of the general theorem (2.8) to be proven below. We note expressly that this relation holds for any portion of a material surface in contact with a perfect fluid in irrotational motion.

A second special case is obtained if, before integration over S, (2.1) is multiplied by $\boldsymbol{a}\cdot\boldsymbol{n}$, where $\boldsymbol{a}$ is an arbitrary vector field. Upon integration over S and use of (2.3) the following equation is obtained:

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_S \phi(\boldsymbol{a}\cdot\boldsymbol{n})\,\mathrm{d}S = \int_S \phi\left\{\boldsymbol{n}\cdot\frac{\mathrm{d}\boldsymbol{a}}{\mathrm{d}t}+\boldsymbol{a}\cdot\left[\frac{\mathrm{d}\boldsymbol{n}}{\mathrm{d}t}-\boldsymbol{n}(\boldsymbol{n}\cdot\boldsymbol{\nabla})(\boldsymbol{u}\cdot\boldsymbol{n})\right]\right\}\mathrm{d}S + \frac{1}{2}\int_S (\boldsymbol{u}\cdot\boldsymbol{u})(\boldsymbol{a}\cdot\boldsymbol{n})\,\mathrm{d}S - \int_S p'(\boldsymbol{a}\cdot\boldsymbol{n})\,\mathrm{d}S. \quad (2.5)$$

This equation can be put into a more useful form by expressing $\mathrm{d}\boldsymbol{n}/\mathrm{d}t$ in terms of the velocity potential ϕ. To this end we make use of the following relation:

$$\frac{\mathrm{d}\boldsymbol{n}}{\mathrm{d}t} = \boldsymbol{n}(\boldsymbol{n}\cdot\boldsymbol{\nabla})(\boldsymbol{u}\cdot\boldsymbol{n})-(\boldsymbol{n}\cdot\boldsymbol{\nabla})\,\boldsymbol{u}, \quad (2.6)$$

which is derived in the Appendix, to find

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_S \phi\,(\boldsymbol{a}\cdot\boldsymbol{n})\,\mathrm{d}s = \int_S \phi\left[\boldsymbol{n}\cdot\frac{\mathrm{d}\boldsymbol{a}}{\mathrm{d}t}-\boldsymbol{a}\cdot(\boldsymbol{n}\cdot\boldsymbol{\nabla})\,\boldsymbol{u}\right]\mathrm{d}S + \frac{1}{2}\int_S (\boldsymbol{u}\cdot\boldsymbol{u})(\boldsymbol{a}\cdot\boldsymbol{n})\,\mathrm{d}S - \int_S p'(\boldsymbol{a}\cdot\boldsymbol{n})\,\mathrm{d}S. \quad (2.7)$$

It may be noted that, with $\boldsymbol{a} = \boldsymbol{n}$ and use of (2.6), this relation reduces to the previous one, (2.4).

Generalizations of these results are now straightforward. Let $T_{ij\ldots k}$ be a component of a tensorial quantity of any order. We multiply the Bernoulli integral by $T_{ij\ldots k}\,n_l$ and use (2.6) to find

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_S \phi T_{ij\ldots k}\,n_l\,\mathrm{d}S = \int_S \phi\left[\frac{\mathrm{d}T_{ij\ldots k}}{\mathrm{d}t}\,n_l - T_{ij\ldots k}(\boldsymbol{n}\cdot\boldsymbol{\nabla})\,\phi_{,l}\right]\mathrm{d}S + \frac{1}{2}\int_S (\boldsymbol{u}\cdot\boldsymbol{u})\,T_{ij\ldots k}\,n_l\,\mathrm{d}S - \int_S p' T_{ij\ldots k}\,n_l\,\mathrm{d}S. \quad (2.8)$$

Here we indicate differentiation with respect to the space coordinates by the standard comma notation of tensor calculus. In particular, selecting $\boldsymbol{a} = \boldsymbol{n}$, and again using (2.6), we find the following generalization of (2.4):

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_S \phi T_{ij\ldots k}\,\mathrm{d}S = \int_S \phi\left[\frac{\mathrm{d}T_{ij\ldots k}}{\mathrm{d}t} - T_{ij\ldots k}(\boldsymbol{n}\cdot\boldsymbol{\nabla})(\boldsymbol{u}\cdot\boldsymbol{n})\right]\mathrm{d}S + \frac{1}{2}\int_S (\boldsymbol{u}\cdot\boldsymbol{u})\,T_{ij\ldots k}\,\mathrm{d}S - \int_S p' T_{ij\ldots k}\,\mathrm{d}S. \quad (2.9)$$

where summation over repeated indices is implied. This result can of course also be proven directly following the steps that leads to (2.4).

The theorem (2.8) can be further generalized to quantities of the type $T_{ij\ldots kl\ldots m}$ $n_n \ldots n_o$. The derivation of this relation is straightforward since it only involves the use of the derivative of a product. However, the final result is somewhat complicated and therefore it will not be given explicitly.

In view of the infinity of relations that can be derived by the method described, the question arises of how many of them are actually independent. Although a complete answer to this question cannot be given at this time, it appears plausible that in principle one should be able to generate as many independent relations as the degrees of freedom of the system considered. Some further comments on this point can be found at the beginning of §5. In this connection it should be noted that the present situation is different from that studied by Benjamin & Olver (1982) who found eight independent conserved integral quantities for two-dimensional surface waves. Indeed, in our case, we are not dealing in general with conserved quantities since the right-hand sides of equations such as (2.8) and (2.9) do not necessarily vanish.

3. A global theorem

The previous results apply to any portion S of a material surface bounding the region occupied by the fluid. In the special case in which S is the entire boundary, one can put the previous results in an alternative 'global' form. Its interest lies in the fact that it will enable us to recover the impulse and virial theorems quoted in §1. Again, in the interest of simplicity, we shall present explicit derivations only for the theorem of (2.7). The general case can be recovered by tracing the same steps starting from (2.9).

Given the complexity of the terms it is best to use index notation. In this notation (2.7) is

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_S \phi a_i n_i \,\mathrm{d}S = \int_S \phi n_i \left[\frac{\mathrm{d}a_i}{\mathrm{d}t} - a_j \phi_{,ij}\right]\mathrm{d}S + \frac{1}{2}\int_S (\phi_{,j}\,\phi_{,j})(a_i n_i)\,\mathrm{d}S + \int_S p'(a_i n_i)\,\mathrm{d}S. \quad (3.1)$$

After a straightforward application of the divergence theorem and some simplifications we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_S \phi a_i n_i \,\mathrm{d}S = \int_V \phi_{,i}\frac{\mathrm{d}a_i}{\mathrm{d}t}\mathrm{d}V + \int_V \phi\left[\left(\frac{\mathrm{d}a_i}{\mathrm{d}t}\right)_{,i} + a_{j,i}\phi_{,ij}\right]\mathrm{d}V + \frac{1}{2}\int_V (\phi_{,j}\,\phi_{,j})\, a_{i,i}\,\mathrm{d}V - \int_S p'(a_i\phi_{,i})\,\mathrm{d}S, \quad (3.2)$$

where V denotes the volume occupied by the fluid. Some simplification can be obtained by recognizing that

$$\frac{\mathrm{d}}{\mathrm{d}t}(a_{j,j}) = \left(\frac{\mathrm{d}a_j}{\mathrm{d}t}\right)_{,j} - a_{j,i}\phi_{,ij}, \quad (3.3)$$

which, upon substitution into (3.2) gives, in vector notation,

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_S \phi \boldsymbol{a}\cdot\boldsymbol{n}\,\mathrm{d}S = \frac{1}{2}\int_V \boldsymbol{u}\cdot\boldsymbol{u}\boldsymbol{\nabla}\cdot\boldsymbol{a}\,\mathrm{d}V + \int_V\left[\frac{\mathrm{d}\boldsymbol{a}}{\mathrm{d}t}\cdot\boldsymbol{\nabla}\phi + \phi\frac{\mathrm{d}}{\mathrm{d}t}(\boldsymbol{\nabla}\cdot\boldsymbol{a})\right]\mathrm{d}V - \int_S p'\boldsymbol{a}\cdot\boldsymbol{n}\,\mathrm{d}S. \quad (3.4)$$

It may be noted that, provided $\boldsymbol{u}\cdot\boldsymbol{u}|\boldsymbol{a}| = o(|\boldsymbol{x}|^{-3})$ at infinity, the surface integrals in this relation need only be extended to the finite portions of the boundary S. Indeed, let S_∞ denote a material surface at infinity. Then, using (2.2), the contribution of S_∞ to this equation is

$$\int_{S_\infty}\left[\frac{\mathrm{d}\phi}{\mathrm{d}t} + p' - \phi(\boldsymbol{n}\cdot\boldsymbol{\nabla})(\boldsymbol{u}\cdot\boldsymbol{n})\right]\boldsymbol{a}\cdot\boldsymbol{n}\,\mathrm{d}S,$$

or, by use of Bernoulli's theorem (2.1),

$$\int_{S_\infty} [\tfrac{1}{2}\boldsymbol{u}\cdot\boldsymbol{u} - \phi(\boldsymbol{n}\cdot\nabla)(\boldsymbol{u}\cdot\boldsymbol{n})]\,\boldsymbol{a}\cdot\boldsymbol{n}\,\mathrm{d}S.$$

The order of magnitude of the two integrands at infinity is the same, and the requirement that the integral vanishes as $|\boldsymbol{x}| \to \infty$ proves the previous statement.

The result (3.4) can be checked with a simpler, alternative derivation. We start with the momentum equation for an inviscid flow,

$$\frac{\mathrm{d}\boldsymbol{u}}{\mathrm{d}t} + \nabla p' = 0. \tag{3.5}$$

We now take the scalar product of this relation with an arbitrary vector field $\boldsymbol{a}(\boldsymbol{x}, t)$ and use $\boldsymbol{u} = \nabla\phi$ to obtain the scalar equation

$$\boldsymbol{a}\cdot\frac{\mathrm{d}\nabla\phi}{\mathrm{d}t} + \boldsymbol{a}\cdot\nabla p' = 0. \tag{3.6}$$

It is possible to convert this equation into a form suitable for application of the divergence theorem by a straightforward manipulation of the first term,

$$\frac{\mathrm{d}}{\mathrm{d}t}(\nabla\cdot\phi\boldsymbol{a}) - \frac{\mathrm{d}\phi}{\mathrm{d}t}\nabla\cdot\boldsymbol{a} - \phi\frac{\mathrm{d}}{\mathrm{d}t}(\nabla\cdot\boldsymbol{a}) - \frac{\mathrm{d}\boldsymbol{a}}{\mathrm{d}t}\cdot\nabla\phi + \boldsymbol{a}\cdot\nabla p' = 0. \tag{3.7}$$

Upon elimination of $\mathrm{d}\phi/\mathrm{d}t$ by use of the Bernoulli integral (2.1) we find

$$\frac{\mathrm{d}}{\mathrm{d}t}(\nabla\cdot\phi\boldsymbol{a}) - \tfrac{1}{2}\boldsymbol{u}\cdot\boldsymbol{u}\nabla\cdot\boldsymbol{a} - \frac{\mathrm{d}\boldsymbol{a}}{\mathrm{d}t}\cdot\nabla\phi - \phi\frac{\mathrm{d}}{\mathrm{d}t}(\nabla\cdot\boldsymbol{a}) + \nabla\cdot(p'\boldsymbol{a}) = 0. \tag{3.8}$$

In contrast to the previous derivations, we now consider a liquid volume $V(t)$ bounded by surfaces, collectively denoted by $\Sigma(t)$, not necessarily material. For any such volume we have the transport theorem

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_V f\,\mathrm{d}V = \int_V \frac{\mathrm{d}f}{\mathrm{d}t}\,\mathrm{d}V + \int_\Sigma f(\boldsymbol{v}-\boldsymbol{u})\cdot\boldsymbol{n}\,\mathrm{d}\Sigma, \tag{3.9}$$

where $\boldsymbol{v}$ is the local surface velocity of the surface Σ. Upon integration over V of (3.8), use of (3.9) and the divergence theorem, one finds

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Sigma \phi\boldsymbol{a}\cdot\boldsymbol{n}\,\mathrm{d}\Sigma = \frac{1}{2}\int_V \boldsymbol{u}\cdot\boldsymbol{u}\nabla\cdot\boldsymbol{a}\,\mathrm{d}V + \int_V\left[\frac{\mathrm{d}\boldsymbol{a}}{\mathrm{d}t}\cdot\nabla\phi + \phi\frac{\mathrm{d}}{\mathrm{d}t}(\nabla\cdot\boldsymbol{a})\right]\mathrm{d}V$$
$$-\int_\Sigma p'\boldsymbol{a}\cdot\boldsymbol{n}\,\mathrm{d}\Sigma + \int_\Sigma [\nabla\cdot(\phi\boldsymbol{a})](\boldsymbol{v}-\boldsymbol{u})\cdot\boldsymbol{n}\,\mathrm{d}\Sigma. \tag{3.10}$$

This relation reduces to (3.4) provided that, as will be assumed henceforth, $\boldsymbol{v}\cdot\boldsymbol{n} = \boldsymbol{u}\cdot\boldsymbol{n}$ over Σ. This will be the case for free surfaces, and also for impermeable surfaces such as those of bodies immersed in the liquid. Similar generalizations are readily obtained for the global forms of (2.8) and (2.9).

4. Some simple special cases

As a first application of our results we now show how the virial and the impulse theorems can be derived from them.

If we take $\boldsymbol{a} = \boldsymbol{x}$, the location of a Lagrangian particle, we have

$$\frac{\mathrm{d}\boldsymbol{x}}{\mathrm{d}t} = \nabla\phi, \tag{4.1}$$

and substitution into (3.4) gives

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_S \phi\boldsymbol{x}\cdot\boldsymbol{n}\,\mathrm{d}S = \frac{5}{2}\int_V \boldsymbol{u}\cdot\boldsymbol{u}\,\mathrm{d}V - \int_S p'\boldsymbol{x}\cdot\boldsymbol{n}\,\mathrm{d}S, \tag{4.2}$$

which is the virial theorem proven by Benjamin (1987) and Longuet-Higgins (1989). The 'local' version of this result can be obtained by choosing, in (2.7), $\boldsymbol{a} = \boldsymbol{x}$ to find

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_S \phi(\boldsymbol{x}\cdot\boldsymbol{n})\,\mathrm{d}S = \int_S \phi[\boldsymbol{n}\cdot\boldsymbol{u} - \boldsymbol{x}\cdot(\boldsymbol{n}\cdot\nabla)\,\boldsymbol{u}]\,\mathrm{d}S + \frac{1}{2}\int_S (\boldsymbol{u}\cdot\boldsymbol{u})\,(\boldsymbol{x}\cdot\boldsymbol{n})\,\mathrm{d}S - \int_S p'(\boldsymbol{x}\cdot\boldsymbol{n})\,\mathrm{d}S. \tag{4.3}$$

Retracing the steps leading from (2.7) to (3.4), the previous form (4.2) is recovered. Equation (4.3) generalizes (4.2) to the case in which S does not constitute the entire (material) finite boundary of the region occupied by the fluid.

To recover the impulse theorem, in (2.9) we choose the tensor $\boldsymbol{T}$ to be an arbitrary constant unit vector e_i. The result is

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_S \phi e_i n_i\,\mathrm{d}S = -\int_S \phi e_i(\boldsymbol{n}\cdot\nabla)\,\phi_{,i}\,\mathrm{d}S + \frac{1}{2}\int_S (\boldsymbol{u}\cdot\boldsymbol{u})\,e_i n_i\,\mathrm{d}S - \int_S p' e_i n_i\,\mathrm{d}S. \tag{4.4}$$

Since e_i is a constant, it can be eliminated and the result written in vectorial notation as

$$\frac{\mathrm{d}\boldsymbol{I}}{\mathrm{d}t} = \int_S (\tfrac{1}{2}\boldsymbol{u}\cdot\boldsymbol{u} - p')\,\boldsymbol{n}\,\mathrm{d}S - \int_S \phi(\boldsymbol{n}\cdot\nabla)\,\boldsymbol{u}\,\mathrm{d}S, \tag{4.5}$$

where

$$\boldsymbol{I} = \int_S \phi\boldsymbol{n}\,\mathrm{d}S. \tag{4.6}$$

If the surface S is closed, this quantity represents the Kelvin impulse of the fluid due to the motion of S. In this case, provided S is simply connected, Blake & Cerone (1982) have proven the identity

$$\int_S (\phi\phi_{,ij} - \tfrac{1}{2}\boldsymbol{u}\cdot\boldsymbol{u})\,n_i\,\mathrm{d}S = \int_S (\tfrac{1}{2}\boldsymbol{u}\cdot\boldsymbol{u} - \phi_{,i}\,\phi_{,j})\,n_i\,\mathrm{d}S. \tag{4.7}$$

This relation holds also if S collectively denotes several closed, simply connected, disjoint surfaces. By making use of it, the previous result (4.5) may be written

$$\frac{\mathrm{d}\boldsymbol{I}}{\mathrm{d}t} = \int_S \{(\boldsymbol{n}\cdot\boldsymbol{u})\,\boldsymbol{u} - [\tfrac{1}{2}\boldsymbol{u}\cdot\boldsymbol{u} - p']\,\boldsymbol{n}\}\,\mathrm{d}S. \tag{4.8}$$

Consider now a situation in which the fluid is bounded by one or more closed finite surfaces S (defining, for example, one or more bubbles) and other boundaries B. We add and subtract to the right-hand side of (4.8) the integral over B of $(\boldsymbol{n}\cdot\boldsymbol{u})\,\boldsymbol{u} - \frac{1}{2}(\boldsymbol{u}\cdot\boldsymbol{u})\,\boldsymbol{n}$. Since, as is easy to prove, the contribution from the surface at infinity vanishes, we can use the divergence theorem to show that the integral over S of this quantity exactly cancels that over B so that we are left with

$$\frac{\mathrm{d}\boldsymbol{I}}{\mathrm{d}t} = \int_B [\tfrac{1}{2}(\boldsymbol{u}\cdot\boldsymbol{u})\,\boldsymbol{n} - (\boldsymbol{n}\cdot\boldsymbol{u})\,\boldsymbol{u}]\,\mathrm{d}S_B - \int_S p'\boldsymbol{n}\,\mathrm{d}S.$$

In the case of a bubble with negligible surface tension effects, p' is a constant and the last integral vanishes. This relation reduces then to the form (1.1) previously quoted.

Another interesting quantity is the moment of impulse (Wu 1976; Blake 1983) defined by

$$\boldsymbol{M} = \int_S \phi\boldsymbol{x}\times\boldsymbol{n}\,\mathrm{d}S. \tag{4.9}$$

By letting, in (2.8), $T_{ij\ldots k} = \epsilon_{jkl}\,x_k$, where ϵ_{ijk} is the alternating tensor, we readily find

$$\frac{\mathrm{d}\boldsymbol{M}}{\mathrm{d}t} = \int_S \phi[\boldsymbol{u}\times\boldsymbol{n} - \boldsymbol{x}\times(\boldsymbol{n}\cdot\boldsymbol{\nabla})\,\boldsymbol{u}]\,\mathrm{d}S + \frac{1}{2}\int_S \boldsymbol{u}\cdot\boldsymbol{u}(\boldsymbol{x}\times\boldsymbol{n})\,\mathrm{d}S - \int_S p'\boldsymbol{x}\times\boldsymbol{n}\,\mathrm{d}S. \tag{4.10}$$

In the special case in which S denotes the complete boundary of the flow, consisting of material or impermeable surfaces, application of the divergence theorem shows the first two integrals in the right-hand side to vanish identically and one is left with

$$\frac{\mathrm{d}\boldsymbol{M}}{\mathrm{d}t} = -\int_S p'\boldsymbol{x}\times\boldsymbol{n}\,\mathrm{d}S, \tag{4.11}$$

from which the theorem of conservation of angular momentum is readily recovered.

The virial, the impulse, and the moment of impulse are linear quantities in ϕ. An obvious nonlinear quantity of interest is the kinetic energy, which we can study by choosing $\boldsymbol{a} = \boldsymbol{\nabla}\phi$ in (3.4). Since in this case $\boldsymbol{\nabla}\cdot\boldsymbol{a} = \nabla^2\phi = 0$, we have from (3.4)

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_S \phi\frac{\partial\phi}{\partial n}\,\mathrm{d}S = \int_V \frac{\mathrm{d}\boldsymbol{\nabla}\phi}{\mathrm{d}t}\cdot\boldsymbol{\nabla}\phi\,\mathrm{d}V - \int_S p'\frac{\partial\phi}{\partial n}\,\mathrm{d}S.$$

It is well known that the integral in the left-hand side is twice the kinetic energy E_{K} of the fluid divided by the density. By use of the momentum equation (3.5), the volume integral can be rewritten as the integral of $\boldsymbol{\nabla}p'\cdot\boldsymbol{\nabla}\phi = \boldsymbol{\nabla}\cdot(p'\boldsymbol{\nabla}\phi)$, which, by the divergence theorem, is seen to equal the last term. The final result is therefore

$$\frac{\mathrm{d}}{\mathrm{d}t}E_{\mathrm{K}} = -\int_S p'\boldsymbol{u}\cdot\boldsymbol{n}\,\mathrm{d}S,$$

which expresses the energy theorem for the fluid under the present hypotheses.

As a final example, we apply (2.4) to the case of linear oscillations of a bubble. In this case the first two terms in the right-hand side vanish and, using angle brackets to denote averages over the bubble surface, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}(S\langle\phi\rangle) = -S\langle p'\rangle. \tag{4.12}$$

For linear oscillations S can be taken out of the time derivative. Upon differentiation with respect to time we then have

$$\frac{\mathrm{d}^2\langle\phi\rangle}{\mathrm{d}t^2} = -\frac{\mathrm{d}p'}{\mathrm{d}V}\frac{\mathrm{d}V}{\mathrm{d}t}, \tag{4.13}$$

where V denotes the bubble volume and p' has been taken to be a (possibly complex) function of V only, as is legitimate in the linear case. Since, however,

$$\frac{\mathrm{d}V}{\mathrm{d}t} = -\int_S \boldsymbol{u}\cdot\boldsymbol{n}\,\mathrm{d}S,$$

with $\mathrm{d}/\mathrm{d}t \rightarrow \mathrm{i}\omega$, where ω is the oscillation frequency of the bubble, we find

$$\omega^2 = -\frac{\mathrm{d}p'}{\mathrm{d}V}\frac{S}{\langle\phi\rangle}\left\langle\frac{\partial\phi}{\partial n}\right\rangle. \tag{4.14}$$

This result is well known (Strasberg 1953). For a recent application see Oguz & Prosperetti (1989).

In the next section we apply the previous theorems to a more complicated problem.

5. Application to the oscillation of two bubbles

As an example of the possible use of the integral theorems derived in §2, and to demonstrate their usefulness, we now apply them to a problem for which earlier results, such as the impulse or virial theorem, can only furnish partial information.

We consider two bubbles in an ambient pressure field p_∞ oscillating with a frequency ω. For analytical convenience we assume that the bubbles remain spherical at all times with radii $R_1(t)$ and $R_2(t)$. This assumption is a valid approximation provided that the distance $D(t)$ between their centres is sufficiently large (Zavtrak 1987). In any event, one may expect the results to have at least a qualitative value even if this condition is not strictly fulfilled for all times. The geometry of the system is depicted in figure 1.

The configuration of this system is specified uniquely by the instantaneous values of the radii and of the positions of the bubbles' centres. One therefore needs the equivalent of eight first-order differential equations to describe the evolution of the system. If shape deformations of the bubbles were allowed, this number would be greater. The fact that our theorems can be applied to the surface of each individual bubble separately, and that as many integral relations as are needed can be generated by considering in (2.9) the surface integrals of $\phi(\boldsymbol{n}\cdot\boldsymbol{x})(\boldsymbol{n}\cdot\boldsymbol{x})\ldots(\boldsymbol{n}\cdot\boldsymbol{x})$ or other expressions, is evidently a very useful feature of those theorems in situations such as this one.

The flip side of the coin is, however, that it is not clear *a priori* how to select, among the infinite possibilities, the equations to use in a particular case. However, one may expect that most equations would contain essentially equivalent information and, provided a sufficient set is chosen in any 'reasonable' way, the choice of any other 'reasonable' set would lead to essentially equivalent results. We shall have further comments on the 'reasonableness' of the choice later on. For the time being, let us just appeal to criteria of simplicity and physical insight.

One may expect that, in a problem such as the present one, the integral statements of the (local) impulse theorem in the direction of the line of centres should contain relevant information on the translatory motion of the bubbles. Accordingly, we generate two second-order equations by applying, to the surface of each bubble, the relations

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_S \phi(\hat{\boldsymbol{z}}\cdot\boldsymbol{n})\,\mathrm{d}S = -\int_S [\hat{\boldsymbol{z}}\cdot(\boldsymbol{u}\cdot\boldsymbol{\nabla})\,\boldsymbol{u}]\,\phi\,\mathrm{d}S + \frac{1}{2}\int_S \boldsymbol{u}\cdot\boldsymbol{u}(\hat{\boldsymbol{z}}\cdot\boldsymbol{n})\,\mathrm{d}S - \int_S p'(\hat{\boldsymbol{z}}\cdot\boldsymbol{n})\,\mathrm{d}S, \tag{5.1}$$

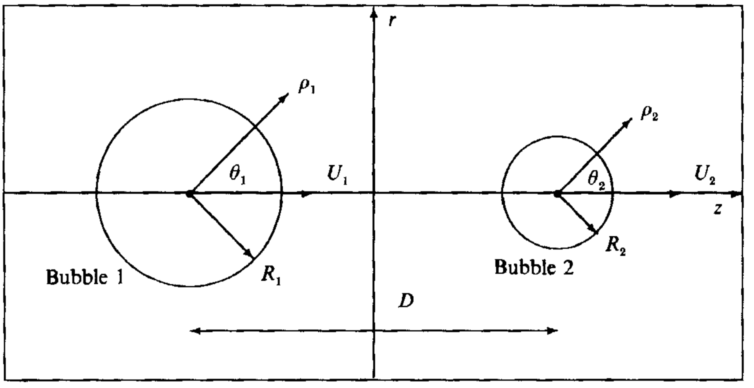

FIGURE 1. Geometry for the bubble-interaction problem.

where $\boldsymbol{z}$ is a unit vector in the positive direction of the z-axis, i.e. the line joining the centres of the bubbles. A simple choice for another pair of equations is (2.4), again applied to the surface of each bubble. Another possibility might be, for example, the (local) virial theorem,

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_S \phi(\boldsymbol{x}\cdot\boldsymbol{n})\,\mathrm{d}S = \int_S \phi[\boldsymbol{u}\cdot\boldsymbol{n}-\boldsymbol{x}\cdot(\boldsymbol{n}\cdot\nabla\boldsymbol{u})]\,\mathrm{d}S+\frac{1}{2}\int_S(\boldsymbol{u}\cdot\boldsymbol{u})\,(\boldsymbol{x}\cdot\boldsymbol{n})\,\mathrm{d}S-\int_S p'\boldsymbol{x}\cdot\boldsymbol{n}\,\mathrm{d}S. \tag{5.2}$$

We have found that the two choices lead to indistinguishable results for the evolution of the system. In applying the previous integral relations, p' will be taken to be constant on the surface of the bubbles. This will strictly be correct in the absence of surface-tension effects. However, in view of the sphericity assumption, it can also be justified when surface tension is retained, as here.

To proceed further we need an *ansatz* on the form of the potential. The simplest possibility is to assume

$$\phi = -\frac{\dot{R}_1 R_1^2}{\rho_1}-\frac{\dot{R}_2 R_2^2}{\rho_2}-\frac{U_1 R_1^3\cos\theta_1}{2\rho_1^2}-\frac{U_2 R_2^3\cos\theta_2}{2\rho_2^2}, \tag{5.3}$$

where ρ_1, θ_1 and ρ_2, θ_2 are local spherical coordinates for the individual bubbles. This form assumes that the jth bubble executes radial pulsations with velocity $\dot{R}_j$, and translates with velocity U_j with respect to the flow induced by the other bubble in its vicinity.

Despite the relatively simple form of the potential, the integrals required are somewhat complex. For this reason we exploit the assumption that $R_1+R_2 \ll D$ to approximate the potential in the neighbourhood of each bubble by a Taylor series truncated to the first order. With this approximation, in the neighbourhood of the first bubble, we have $\phi \approx \phi_1$ with

$$\phi_1 = -\frac{\dot{R}_1 R_1^2}{\rho_1}-\frac{U_1 R_1^3\cos\theta_1}{2\rho_1^2}-\frac{\dot{R}_2 R_2^2}{D}+\frac{U_2 R_2^3\cos\theta_2}{2D^2}-\left(\frac{\dot{R}_2 R_2^2}{D^2}-\frac{U_2 R_2^3\cos\theta_2}{D^3}\right)\rho_1\cos\theta_1, \tag{5.4}$$

and similarly, near the second bubble, $\phi \approx \phi_2$ with

$$\phi_2 = -\frac{\dot{R}_2 R_2^2}{\rho_2}-\frac{U_2 R_2^3\cos\theta_2}{2\rho_2^2}-\frac{\dot{R}_1 R_1^2}{D}-\frac{U_1 R_1^3\cos\theta_1}{2D^2}+\left(\frac{\dot{R}_1 R_1^2}{D^2}+\frac{U_1 R_1^3\cos\theta_1}{D^3}\right)\rho_2\cos\theta_2. \tag{5.5}$$

It should be noted that these relations are not power series expansions in $1/D$ of an exact potential since, for example, the quadrupole terms arising from surface deformation have been neglected in (5.3), and such terms would give further contributions of order $1/D^3$. Rather, the underlying assumption is that the higher-order multipoles contribute little to the interaction between the spheres. Inspection of (5.4) and (5.5) shows that ϕ_1 and ϕ_2 both have the structure

$$\phi = A+\frac{B}{\rho}+C\rho\cos\theta+\frac{E\cos\theta}{\rho^2}, \tag{5.6}$$

where ρ and θ are local spherical coordinates and, for bubble 1,

$$A_1 = -\frac{\dot{R}_2 R_2^2}{D}+\frac{U_2 R_2^3}{2D^2}, \tag{5.7}$$

$$B_1 = -\dot{R}_1 R_1^2, \tag{5.8}$$

$$C_1 = -\frac{\dot{R}_2 R_2^2}{D^2}+\frac{U_2 R_2^3}{D^3}, \tag{5.9}$$

$$E_1 = -\tfrac{1}{2}U_1 R_1^3, \tag{5.10}$$

while, for bubble 2,

$$A_2 = -\frac{\dot{R}_1 R_1^2}{D}-\frac{U_1 R_1^3}{2D^2}, \tag{5.11}$$

$$B_2 = -\dot{R}_2 R_2^2, \tag{5.12}$$

$$C_2 = \frac{\dot{R}_1 R_1^2}{D^2}+\frac{U_1 R_1^3}{D^3}, \tag{5.13}$$

$$E_2 = -\tfrac{1}{2}U_2 R_2^3. \tag{5.14}$$

It is clear from (5.6) that the translational velocities relative to the absolute frame are $U+C$, so that the relative distance D changes according to the relation

$$\dot{D} = U_2+\frac{\dot{R}_1 R_1^2}{D^2}+\frac{U_1 R_1^3}{D^3}-U_1+\frac{\dot{R}_2 R_2^2}{D^2}-\frac{U_2 R_2^3}{D^3}. \tag{5.15}$$

To close the system we need equations for $\dot{R}_1, \dot{R}_2, \dot{U}_1, \dot{U}_2$, which will be derived in the manner previously explained. The impulse equation (5.1) gives, for each bubble,

$$\frac{\mathrm{d}}{\mathrm{d}t}[\tfrac{1}{3}(CR^3+E)] = -BC. \tag{5.16}$$

Owing to the assumed uniformity of p', this quantity drops out from this equation as expected. Bassett (1888) treats the case in which the two spheres are fixed so that $U+C=0$. Using this relation, it is easy to see that (5.16) coincides with his result in Art. 241 up to terms of order D^{-4} included. As for the terms of order D^{-5}, as a consequence of the neglect of quadrupole contributions in the form (5.3) of the potential, our expression is only partially correct.

Again for each bubble, (2.4) gives, after some straightforward but lengthy algebra,

$$\frac{\mathrm{d}}{\mathrm{d}t}[2(AR^2+BR)]+R^2\left(\frac{4BA}{R^3}+\frac{3B^2}{R^4}-C^2+\frac{2E^2}{R^6}+\frac{4CE}{R^3}+2p'\right) = 0. \tag{5.17}$$

Use of the theorem (5.2) leads instead to

$$\frac{\mathrm{d}}{\mathrm{d}t}[2(AR^3+BR^2)]+R^3\left(\frac{4BA}{R^3}+\frac{3B^2}{R^4}-C^2+\frac{2E^2}{R^6}+\frac{4CE}{R^3}+2p'\right)$$
$$+R^2\left(\frac{2B^2}{R}+2AB-\tfrac{2}{3}C^2R^3+\frac{4}{3}\frac{E^2}{R^3}+\tfrac{2}{3}CE\right)=0. \quad (5.18)$$

Although for numerical work it might be preferable to use AR^2+BR and CR^3+E as auxiliary dependent variables, we give here explicit expressions for the time derivatives of these quantities. From the definitions (5.7)–(5.14) we find, for bubble 1,

$$\frac{\mathrm{d}}{\mathrm{d}t}[A_1R_1^2+B_1R_1]=-\frac{\dot{R}_2R_2^2R_1^2}{D}\left(\frac{\ddot{R}_2}{\dot{R}_2}+2\frac{\dot{R}_2}{R_2}+2\frac{\dot{R}_1}{R_1}-\frac{\dot{D}}{D}\right)$$
$$+\frac{U_2R_2^3R_1^2}{2D^2}\left(\frac{\dot{U}_2}{U_2}+3\frac{\dot{R}_2}{R_2}+2\frac{\dot{R}_1}{R_1}-2\frac{\dot{D}}{D}\right)-\dot{R}_1R_1^3\left(\frac{\ddot{R}_1}{\dot{R}_1}+3\frac{\dot{R}_1}{R_1}\right), \quad (5.19)$$

$$\frac{\mathrm{d}}{\mathrm{d}t}[C_1R_1^3+E_1]=-\frac{\dot{R}_2R_2^2R_1^3}{D^2}\left(\frac{\ddot{R}_2}{\dot{R}_2}+2\frac{\dot{R}_2}{R_2}+3\frac{\dot{R}_1}{R_1}-2\frac{\dot{D}}{D}\right)$$
$$+\frac{U_2R_2^3R_1^3}{D^3}\left(\frac{\dot{U}_2}{U_2}+3\frac{\dot{R}_2}{R_2}+3\frac{\dot{R}_1}{R_1}-3\frac{\dot{D}}{D}\right)-\tfrac{1}{2}U_1R_1^3\left(\frac{\dot{U}_1}{U_1}+3\frac{\dot{R}_1}{R_1}\right), \quad (5.20)$$

and, for bubble 2,

$$\frac{\mathrm{d}}{\mathrm{d}t}[A_2R_2^2+B_2R_2]=-\frac{\dot{R}_1R_1^2R_2^2}{D}\left(\frac{\ddot{R}_1}{\dot{R}_1}+2\frac{\dot{R}_1}{R_1}+2\frac{\dot{R}_2}{R_2}-\frac{\dot{D}}{D}\right)$$
$$-\frac{U_1R_1^3R_2^2}{2D^2}\left(\frac{\dot{U}_1}{U_1}+3\frac{\dot{R}_1}{R_1}+2\frac{\dot{R}_2}{R_2}-2\frac{\dot{D}}{D}\right)-\dot{R}_2R_2^3\left(\frac{\ddot{R}_2}{\dot{R}_2}+3\frac{\dot{R}_2}{R_2}\right), \quad (5.21)$$

$$\frac{\mathrm{d}}{\mathrm{d}t}[C_2R_2^3+E_2]=-\frac{\dot{R}_1R_1^2R_2^3}{D^2}\left(\frac{\ddot{R}_1}{\dot{R}_1}+2\frac{\dot{R}_1}{R_1}+3\frac{\dot{R}_2}{R_2}-2\frac{\dot{D}}{D}\right)$$
$$+\frac{U_1R_1^3R_2^3}{D^3}\left(\frac{\dot{U}_1}{U_1}+3\frac{\dot{R}_1}{R_1}+3\frac{\dot{R}_2}{R_2}-3\frac{\dot{D}}{D}\right)-\tfrac{1}{2}U_2R_2^3\left(\frac{\dot{U}_2}{U_2}+3\frac{\dot{R}_2}{R_2}\right). \quad (5.22)$$

By using these expressions in the left-hand sides of (5.16) and (5.17) a linear system of four equations in the four quantities $\ddot{R}_1$, $\ddot{R}_2$, $\dot{U}_1$ and $\dot{U}_2$ is obtained. One may conjecture that, if the choice of integral theorems used to obtain the equations is 'reasonable', this system can be solved. Presumably this circumstance indicates that the information included in the equations is sufficient and not redundant. By the same token, if the system turns out to have a singular or nearly-singular matrix, a different set of equations should be used.

Since the process used to derive the above dynamical equations is somewhat formal, it is interesting to examine them in physical terms. A more transparent form of (5.16) obtained from the impulse theorem is

$$\frac{\mathrm{d}}{\mathrm{d}t}[\tfrac{1}{2}\tfrac{4}{3}\pi R_1^3U_1]=\tfrac{4}{3}\pi R_1^3\frac{\mathrm{d}}{\mathrm{d}t}\left(-\frac{\dot{R}_2R_2^2}{D^2}+\frac{U_2R_2^3}{D^3}\right) \quad (5.23)$$

for bubble 1 and

$$\frac{\mathrm{d}}{\mathrm{d}t}[\tfrac{1}{2}\tfrac{4}{3}\pi R_2^3U_2]=-\tfrac{4}{3}\pi R_2^3\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\dot{R}_1R_1^2}{D^2}+\frac{U_1R_1^3}{D^3}\right) \quad (5.24)$$

for bubble 2. These equations are in form which is directly comparable to the equation of motion for a body of volume v and mass m immersed in a flow with velocity $\boldsymbol{V}$ (see e.g. Landau & Lifshitz 1959, section I.11):

$$\frac{\mathrm{d}}{\mathrm{d}t} m U_i = \rho v \frac{\mathrm{d}V_i}{\mathrm{d}t} - \frac{\mathrm{d}}{\mathrm{d}t} M_{ik}(U_k - V_k),$$

where $\boldsymbol{U}$ is the velocity of the body and M_{ik} is the added-mass tensor. Indeed, $m = 0$ for the present case and

$$V_1 = -\frac{\dot{R}_2 R_2^2}{D^2} + \frac{U_2 R_2^3}{D^3}, \tag{5.25}$$

$$V_2 = \frac{\dot{R}_1 R_1^2}{D^2} + \frac{U_1 R_1^3}{D^3}, \tag{5.26}$$

are, for each bubble, the velocity of the flow induced by the other bubble in its vicinity.

The full form of the other equations is somewhat complex and to interpret them it is easier to retain only terms in $1/D$ and to consider the linearized form. In this limit, the two sets of equations coincide and reduce to

$$R_1 \ddot{R}_1 = p_1' - \frac{R_2^2}{D} \ddot{R}_2 \tag{5.27}$$

for bubble 1 and

$$R_2 \ddot{R}_2 = p_2' - \frac{R_1^2}{D} \ddot{R}_1 \tag{5.28}$$

for bubble 2. These can be recognized as the linearized forms of the Rayleigh–Plesset equations adjusted for the pressure field induced by the other bubble at the centre of each bubble (Zabolotskaya 1984).

The fact that the linearized forms of (5.17) and (5.18) coincide suggests that these equations essentially contain the same information and therefore cannot be used together. Their selection as a fundamental set of dynamical equations would not be 'reasonable' in the sense previously explained. An explicit expression for their difference in the fully nonlinear case can be found by multiplying (5.17) by R and subtracting from (5.18) to find

$$-\tfrac{4}{3}\pi R_1^3[(V_2 - \tfrac{1}{2}U_1)(V_2 + U_1)] \tag{5.29}$$

for bubble 1, and

$$-\tfrac{4}{3}\pi R_2^3[(V_1 - \tfrac{1}{2}U_2)(V_1 + U_2)] \tag{5.30}$$

for bubble 2. The difference is zero if

$$V_2 = \tfrac{1}{2}U_1 \quad \text{or} \quad V_2 = -U_1, \quad \text{and} \quad V_1 = \tfrac{1}{2}U_2 \quad \text{or} \quad V_1 = -U_2. \tag{5.31}$$

Interestingly, the first and the third conditions are actually met in the linearized approximation in which the impulse equations (5.23), (5.24) become

$$\tfrac{1}{2}\dot{U}_1 = -\frac{\ddot{R}_2 R_2^2}{D^2} = \dot{V}_2$$

for bubble 1 and

$$\tfrac{1}{2}\dot{U}_2 = \frac{\ddot{R}_1 R_1^2}{D^2} = \dot{V}_1$$

for bubble 2.

After explicit expressions for $\ddot{R}_1$, $\ddot{R}_2$, $\dot{U}_1$ and $\dot{U}_2$ have been obtained by solving the linear system involving these quantities mentioned above, any standard technique can be used to integrate in time. But before this can be done, a definite model for the pressure term p' must be specified. By definition $p' = (p-p_\infty)/\rho$. For simplicity we take the gas in the bubbles to compress adiabatically so that, on the outer surface of the jth bubble,

$$p = p_j^0\left(\frac{R_j^0}{R_j}\right)^{3\gamma} - \frac{2\sigma}{R_j} - 4\mu\frac{\dot{R}_j}{R_j}, \quad j = 1, 2. \tag{5.32}$$

Here σ is the surface tension, μ the viscosity, and γ the adiabatic index. The superscript 0 indicates equilibrium conditions. Since the velocity of the radial motion is typically much larger than that of the translatory motion of the bubbles, it is not inconsistent to retain viscous effects in (5.23), although no drag terms appear in the momentum equations. For the ambient pressure p_∞ we take

$$p_\infty = p_s(1-\epsilon \sin \omega t), \tag{5.33}$$

where p_s is the static pressure, and

$$p_j^0 = p_s + \frac{2\sigma}{R_j^0}$$

is the pressure inside each bubble at equilibrium.

In the numerical calculations we use non-dimensional quantities in which the reference length and time are R_1^0 and ω^{-1} respectively, and the reference pressure is $\rho(\omega R_1^0)^2$. In these dimensionless variables, indicated by an asterisk, we have

$$p_j'^* = \frac{p_1^0}{\rho(\omega R_1^0)^2}\left[\frac{p_j^0}{p_1^0}\left(\frac{R_j^0/R_1^0}{R_j^*}\right)^{3\gamma} - \frac{W_j(R_j^0/R_1^0) + M_j \dot{R}_j^*}{R_j^*} - p_{\infty j}^*\right],$$

where $$p_{\infty j}^* = (1-W_j)(1-\epsilon \sin t_*), \quad W_j = \frac{2\sigma}{p_j^0 R_j^0}, \quad M_j = \frac{4\omega\mu}{p_j^0}. \tag{5.34}$$

We present some numerical results obtained from the previous system of equations in the next section.

6. Numerical examples

The study of the mutual interaction between two linearly oscillating bubbles has a long history that goes back to C. A. Bjerknes in 1868. The linear theory is very well understood (Crum 1975; Apfel 1976; Weiser, Apfel & Neppiras 1984; Prosperetti 1984) and leads to the following expression for the force F_2 exerted by bubble 1 on bubble 2 in the direction of the line of centres:

$$F_2 = -\frac{\rho\omega^2 V_1^0 V_2^0}{8\pi D^2}\delta_1 \delta_2 \cos(\psi_1 - \psi_2). \tag{6.1}$$

Here V_j^0 is the equilibrium volume of the bubbles, δ_j is the amplitude of the volume pulsation so that $V_j(t) = V_j^0[1-\delta_j \sin(\omega t + \psi_j)]$, and ψ_j is the phase of the bubble oscillation with respect to the driving sound field. A negative force implies attraction and a positive one repulsion. In the linear regime of oscillation, this equation shows that two bubbles both driven below or above resonance will attract, while if one is below and the other above resonance, the force will be repulsive. As a simple application of the equations derived in the previous section we briefly examine here the validity of this statement for the nonlinear case.

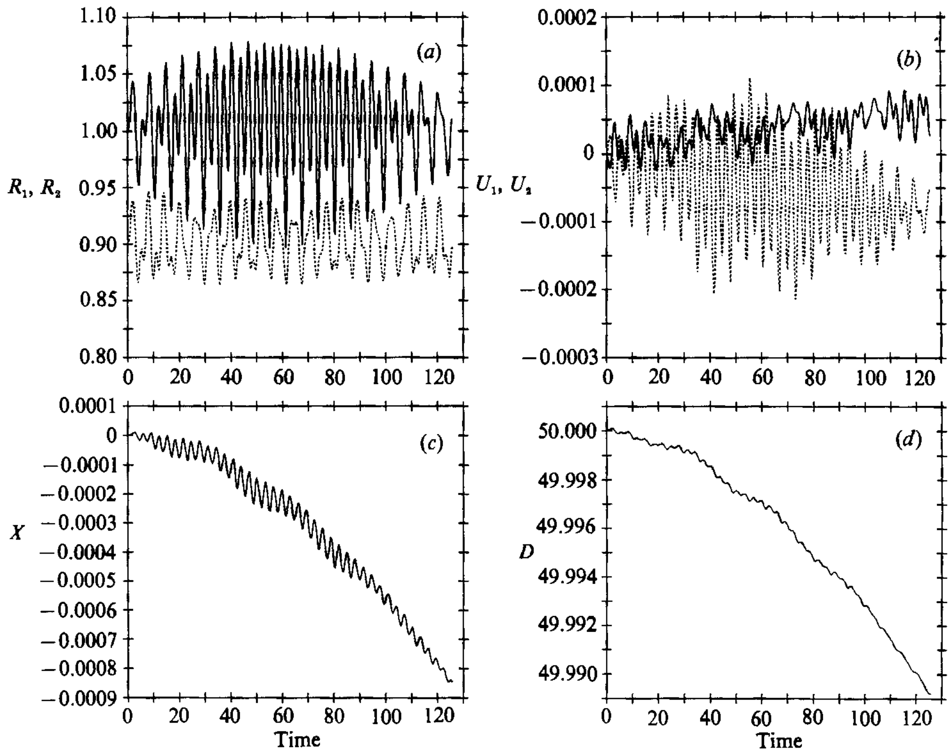

FIGURE 2. (*a*) The dimensionless radius versus time for two oscillating bubbles; and (*b*) the time variation of U_1 and U_2 defined in equation (5.3). The continuous lines are for bubble 1 and the dashed lines for bubble 2. (*c*) The position X of the midpoint between the bubbles, and (*d*) the distance D between the centres, both in units of R_1^0. Here the equilibrium radii are $R_1^0 = 0.1$ mm and $R_2^0 = 0.09$ mm. The initial distance is $D(0) = 5$ mm, the frequency ω of the driving sound field is such that $\omega/\omega_1^0 = 0.51$, where ω_1^0 is the linear resonance frequency of bubble 1. The dimensionless sound amplitude is $\epsilon = 0.1$.

Figures 2 and 3 refer to two slightly different bubbles, having equilibrium radii of 0.1 and 0.09 mm, initially at a distance $D(0) = 5$ mm, and driven at a frequency $\omega = 0.51\omega_1^0$, where ω_1^0 is the linear resonance frequency of the bigger bubble which, according to the well-known formula (Minnaert 1933)

$$(\omega_1^0)^2 = \frac{1}{\rho(R_1^0)^2}\left(3\gamma p_1^0 - \frac{2\sigma}{R_1^0}\right), \tag{6.2}$$

has the value $\omega_1^0/2\pi = 33$ kHz, approximately, in a static pressure p_s of 1 bar. Since the ratio ω/ω_2^0 has the value 0.46, both bubbles are driven below resonance and, according to linear theory, should therefore attract. This is indeed seen to be the case at low forcing, $\epsilon = 0.1$, in figure 2. In this and in the following figures panel (*a*) shows $R_1(t)$ and $R_2(t)$ and panel (*b*) shows $U_1(t)$ and $U_2(t)$. In both panels the continuous line is for bubble 1 and the dotted line for bubble 2. Panel (*c*) shows the position $X(t)$ of the midpoint of the segment joining the centres, and panel (*d*) is a graph of the relative distance $D(t)$, both non-dimensionalized by R_1^0. The time is given in units of ωt. It can be seen from the second panel that the bigger bubble has a strong effect on the translation of the smaller bubble so that the centre of the system moves in the direction of the bigger bubble. At a higher forcing, $\epsilon = 0.5$, figure 3, the force is

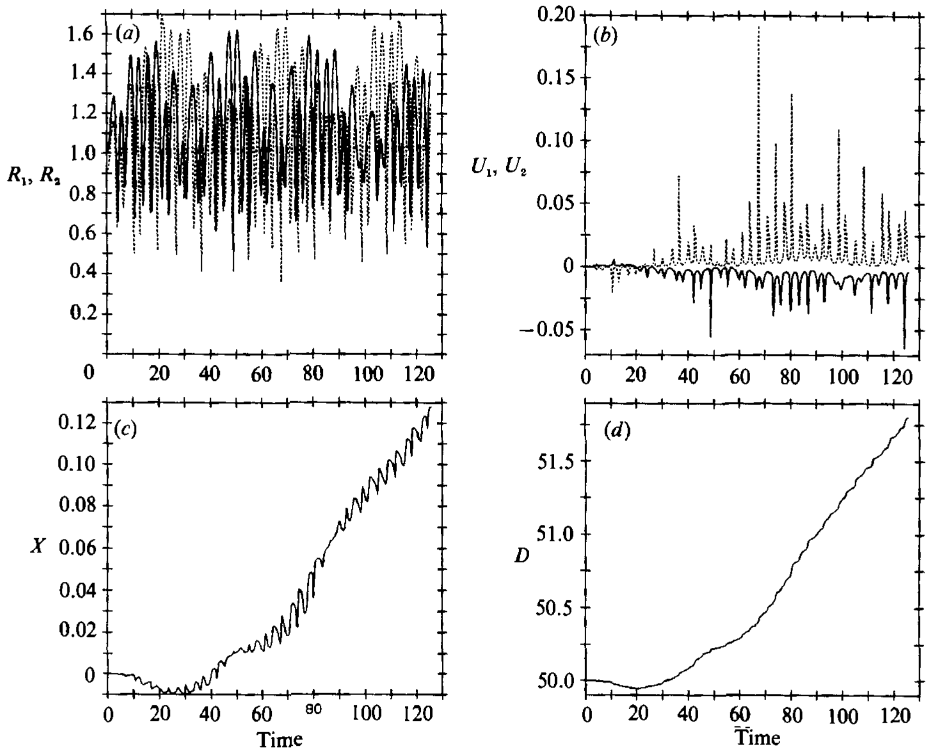

FIGURE 3. As figure 2, with $\epsilon = 0.5$.

essentially always repulsive, in marked contrast with the predictions of linear theory. This behaviour is, however, strongly dependent on the driving frequency. For example, the same two bubbles driven at $\omega/\omega_1^0 = 0.5$, $\omega/\omega_2^0 = 0.4$, for $\epsilon = 0.25$, are found to attract. Our preliminary computations have revealed several intricate features of this system which we shall address in a separate paper.

The existence of repulsive forces in bubbles driven below resonance has been reported earlier by Zabolotskaya (1984) in the linear case. There the effect was due to the change in the oscillation frequency of the bubbles caused by their mutual interaction as obtained from (5.27) and (5.28). Here, the result is instead a clear consequence of nonlinear effects. This finding may have a bearing on some intriguing experimental observations reported by Crum & Nordling (1972) on the motion of bubble clusters in a high-intensity ultrasonic field. They found that the clusters would retain their identity over several hundreds of acoustic periods before being annihilated, a result evidently incompatible with the linear theory of Bjerknes forces. An explanation of our finding could be that, near the first nonlinear resonance, the bubble radius contains a strong component at twice the driving frequency. Owing to the slight difference in bubble radii, this component is driven below resonance for the smaller bubble and above resonance for the bigger one. These components would therefore tend to give rise to a repulsive force. Since, according to (6.1), the magnitude of the force is proportional to the square of the frequency, this force could be more intense than that due to the fundamental. Of course, as shown by Thompson (see e.g. Bassett 1888), a repulsive force can also arise from the oscillatory translational motion of the bubbles. The exact theory shows this effect to be of order

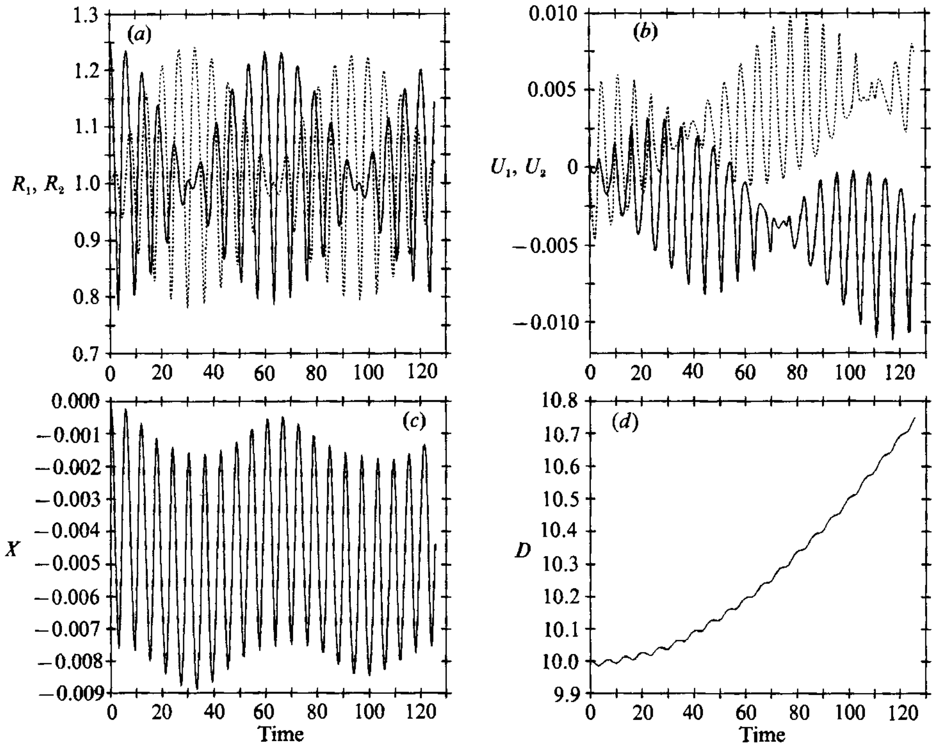

FIGURE 4. As figure 2, but for a case of free oscillations and equal equilibrium radii $R_1^0 = R_2^0 = 0.1$ mm. At $t = 0$, $R_1(0)/R_1^0 = 1.25$, while $R_2(0) = R_2^0$.

D^{-7}. Our approximation might lead to an effect of order D^{-5} due to unbalanced terms of this magnitude. In either case, the distance between centres in these examples is so large that these repulsive forces must be negligible.

Let us now turn to the case of free oscillations, for which $\epsilon = 0$ in (5.34). In figure 4 we consider two equal bubbles with $R^0 = 0.1$ mm, initially at a distance $D(0) = 10R^0 = 1$ mm from each other. Here the time is units of $\omega_1^0 t$. The initial condition for bubble 1 is $R_1(0)/R_1^0 = 1.25$, while bubble 2 is in equilibrium. Like any other system of nearly tuned coupled oscillators the two bubbles exchange their energy at a modulation frequency lower than their oscillation frequency. The force here is weakly repulsive, an indication of the presence of nonlinear effects since the linear phase difference for these initial conditions would be $\frac{1}{2}\pi$ and the resulting force 0 according to (6.1). The mean position also executes modulated oscillations with a non-zero mean component, another indication of the importance of nonlinear effects. For the same situation, when the second bubble is made slightly smaller, $R_2^0 = 0.09$ mm, figure 5, the force becomes strongly attractive, in qualitative agreement with linear theory.

In figure 6 the two bubbles have the same radius, 0.1 mm, and are stated with the same over-expansion $R(0)/R^0 = 1.25$. By images, this situation can also be related to the oscillations of a bubble in the neighbourhood of a rigid wall. The force is strongly attractive and, although the theory is probably quantitatively unreliable when the bubbles become too close, the results suggest coalescence in about ten cycles. The mean position remains fixed at 0, which gives an indication of the accuracy of the numerical procedure.

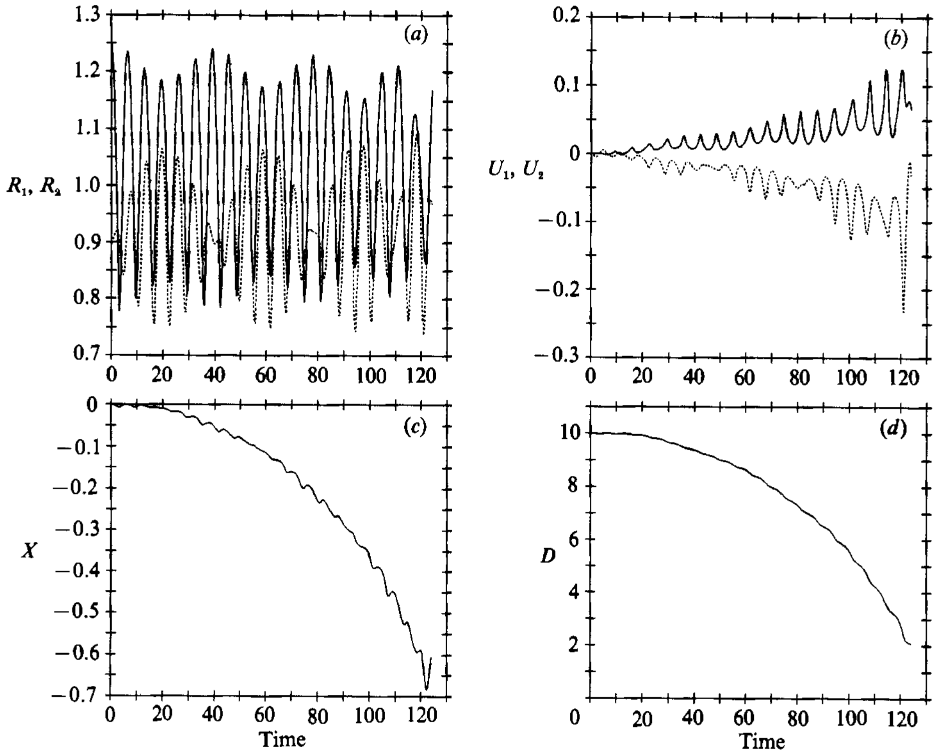

FIGURE 5. As figure 4, but for $R_2^0 = 0.09$ mm.

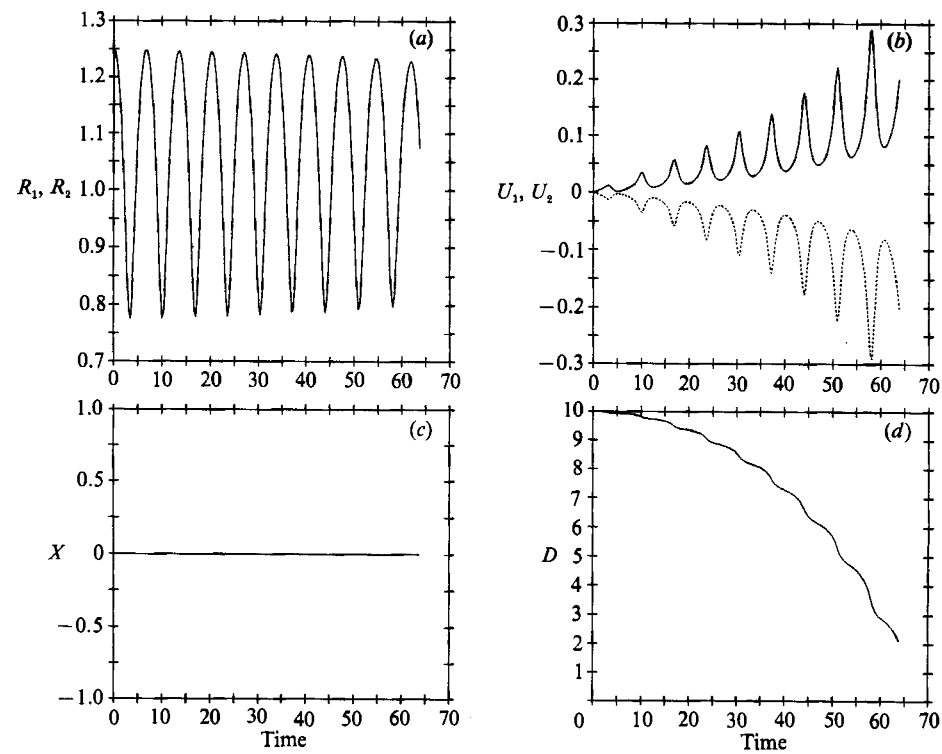

FIGURE 6. As figure 4, but for equal initial conditions $R_1(0)/R_1^0 = R_2(0)/R_2^0 = 1.25$. The two curves in (a) superpose and are indistinguishable as expected.

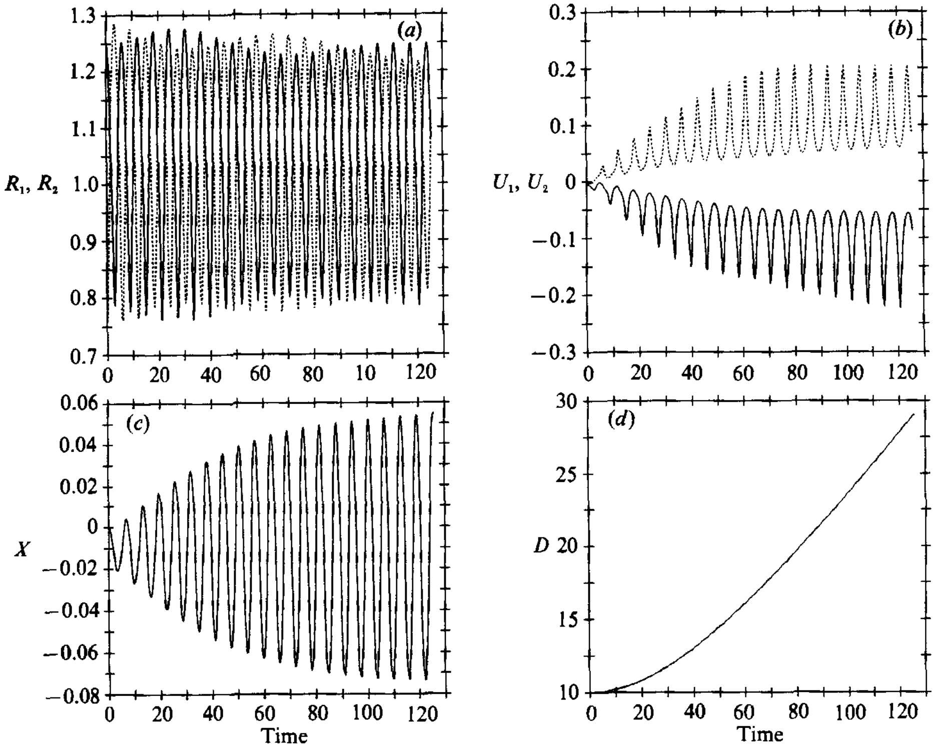

FIGURE 7. As figure 6, but for initial conditions $R_1(0)/R_1^0 = 1.25$, $R_2(0)/R_2^0 = 0.75$.

In our final example, shown in figure 7, the bubbles have again the same equilibrium radius of 0.1 mm, but are started with opposite phases, $R_1(0)/R_1^0 = 1.25$, $R_2(0)/R_2^0 = 0.75$. This is close to simulating the oscillations of a bubble near a plane pressure release boundary, for which however the initial changes in the volume, rather than the radius, should be equal. Here the force is repulsive, as expected.

7. Summary

In the first part of the paper we have derived a very general class of theorems concerning certain integrals of the velocity potential over flow boundaries. It has been shown that the known forms of the impulse and virial theorems are special cases of our general result. Unlike those theorems, however, the integrals appearing in our result do not need to extend over the entire flow boundary, but are applicable to any portion of it.

The theorems obtained appear to be especially useful for approximate calculations, in a manner similar to that of Blake and co-workers who showed how such approaches can provide a correct global picture without the costly calculations of an exact formulation. Since our class of theorems is in principle infinite, again contrary to earlier results, it appears plausible that our formulation enables one to derive as many independent equations as there are degrees of freedom in the approximate formulation.

As an example of the application of our results we have studied the mutual interaction of two bubbles executing forced and free oscillations. An unexpected result of this application has been that nonlinear effects can influence the interaction so strongly as to change the sign of the force with respect to the prediction of the linear theory. This finding may explain the observed persistence of stable bubble clusters in strong acoustic cavitation.

The authors wish to thank Dr Pedro Ponte Castañeda and Dr John Blake for some helpful suggestions. This study has been reported by the Ocean Acoustics Program of ONR.

Appendix

Here we give a simple derivation of (2.6) for the time derivative of the unit normal to a material surface. The transport theorem with $\nabla \cdot \boldsymbol{u} = 0$ is given by (see e.g. Malvern 1969)

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_S f\boldsymbol{n}\,\mathrm{d}S = \int_S \frac{\mathrm{d}f}{\mathrm{d}t}\boldsymbol{n}\,\mathrm{d}S - \int_S (\boldsymbol{n}\cdot\nabla) f\boldsymbol{u}\,\mathrm{d}S, \tag{A 1}$$

where f is an arbitrary function. The above equation (A 1) can also be written as

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_S f\boldsymbol{n}\,\mathrm{d}S = \int_S \frac{\mathrm{d}f}{\mathrm{d}t}\boldsymbol{n}\,\mathrm{d}S + \int_S f\frac{\mathrm{d}\boldsymbol{n}}{\mathrm{d}t}\,\mathrm{d}S - \int_S f\boldsymbol{n}(\boldsymbol{n}\cdot\nabla)(\boldsymbol{u}\cdot\boldsymbol{n})\,\mathrm{d}S, \tag{A 2}$$

in view of the identity (2.3). By taking the difference of (A 1) and (A 2), and in view of the fact that the surface S is arbitrary and can be taken as an arbitrarily small portion of the material surface, we conclude that

$$\frac{\mathrm{d}\boldsymbol{n}}{\mathrm{d}t} = \boldsymbol{n}(\boldsymbol{n}\cdot\nabla)(\boldsymbol{u}\cdot\boldsymbol{n}) - (\boldsymbol{n}\cdot\nabla)\boldsymbol{u}. \tag{A 3}$$

It must also be possible to prove this result directly from considerations of differential geometry. The proof given above, however, is more readily accessible to people trained in fluid mechanics.

REFERENCES

Apfel, R. E. 1976 Technique for measuring the adiabatic compressibility, density, and sound speed of sub-microliter liquid samples. *J. Acoust. Soc. Am.* **59**, 339–343.

Aris, R. 1962 *Vectors, Tensors, and the Basic Equations of Fluid Mechanics*, ch. 9. Prentice Hall.

Bassett, A. B. 1888 *A Treatise on Hydrodynamics*, vol. I, p. 248. London: Deighton, Bell and Co. (Reprinted by Dover, New York, 1961.)

Benjamin, T. B. 1987 Hamiltonian theory for motions of bubbles in an infinite liquid. *J. Fluid Mech.* **181**, 349–379.

Benjamin, T. B. 1989 Note on shape oscillations of bubbles. *J. Fluid Mech.* **203**, 419–424.

Benjamin, T. B. & Ellis, A. T. 1966 The collapse of cavitation bubbles and the pressures thereby produced against solid boundaries. *Phil. Trans. R. Soc. Lond.* A **260**, 221–240.

Benjamin, T. B. & Olver, P. J. 1983 Hamiltonian structure, symmetries and conservation laws for water waves. *J. Fluid Mech.* **125**, 137–85.

Blake, J. R. 1983 The Kelvin impulse: application to bubble dynamics. In *Proc. 8th Austral. Fluid Mech. Conf. Newcastle*, pp. 10.B.1–10.B.4.

Blake, J. R. 1988 The Kelvin impulse: application to cavitation bubble dynamics. *J. Austral. Math. Soc.* B **30**, 127–46.

Blake, J. R. & Cerone, P. 1982 A note on the impulse due to a vapour bubble near a boundary. *J. Austral. Math. Soc.* B **23**, 383–393.

Blake, J. R. & Gibson, D. C. 1981 Growth and collapse of a vapour cavity near a free surface. *J. Fluid Mech.* **111**, 123–140.

Blake, J. R. & Gibson, D. C. 1987 Cavitation bubbles near boundaries. *Ann. Rev. Fluid Mech.* **19**, 99–123.

Blake, J. R., Taib, B. B. & Doherty, G. 1986 Transient cavities near boundaries. Part 1. Rigid boundary. *J. Fluid Mech.* **170**, 479–497.

Crum, L. A. 1975 Bjerknes forces on bubbles in a stationary sound field. *J. Acoust. Soc. Am.* **57**, 1363–1370.

Crum, L. A. & Nordling, D. A. 1972 Velocity of transient cavities in an acoustic stationary wave. *J. Acoust. Soc. Am.* **52**, 294–301.

Kemmer, N. 1977 *Vector Analysis*, ch. 10. Cambridge University Press.

Landau, L. D. & Lifshitz, E. M. 1959 *Fluid Mechanics*, pp. 26–30. Pergamon.

Longuet-Higgins, M. S. 1989 Some integral theorems relating to the oscillation of bubbles. *J. Fluid Mech.* **204**, 159–166.

Malvern, L. E. 1969 *Introduction to the Mechanics of a Continuous Medium*, pp. 210–213. Prentice-Hall.

Minnaert, M. 1933 On musical air bubbles and the sounds of running water. *Phil. Mag.* **16**, 235–248.

Oguz, H. N. & Prosperetti, A. 1989 Oscillations of a bubble in the vicinity of a nearly plane pressure-release boundary. *J. Acoust. Soc. Am.*, in press.

Prosperetti, A. 1979 Boundary conditions at a liquid–vapour interface. *Meccanica* **14**, 34–46.

Prosperetti, A. 1984 Bubble phenomena in sound fields: part two. *Ultrasonics* **22**, 115–23.

Strasberg, M. 1953 The pulsation frequency of nonspherical gas bubbles in liquids. *J. Acoust. Soc. Am.* **25**, 536–7.

Weiser, M. A. H., Apfel, R. E. & Neppiras, E. A. 1984 Interparticle forces on red cells in a standing wave field. *Acustica* **56**, 114–119.

Wu, T. Y. 1976 The momentum theorem for a deformable body in a perfect fluid. *Schiffstechnik* **23**, 229–232.

Zabolotskaya, E. A. 1984 Interaction of gas bubbles in a sound field. *Sov. Phys. Acoust.* **30**, 365–368.

Zavtrak, S. T. 1987 Bjerknes interaction force of two gas bubbles in a sound field. *Sov. Phys. Acoust.* **33**, 145–147.